Midterm Examination

1. Find the following limits:

(a) $\lim_{x \to -4} \frac{x^2 - 6x + 8}{x - 4}$

$\lim_{x \to -4} \frac{(x-2)(x-4)}{x-4} = \lim_{x \to -4} (x-2) = -6$

(unless there was a type in the problem, you can just plug in -4)

(b) $\lim_{x \to -1} \frac{x \sin(x+1) + \sin(x+1)}{x^2 + 2x + 1}$

$= \lim_{x \to -1} \frac{(x+1)\sin(x+1)}{(x+1)(x+1)} = \lim_{x \to -1} \frac{\sin(x+1)}{x+1}$

$= \lim_{x \to -1} \cos(x+1)$ (by L'Hospital's Rule)

=1

(Let me know if you haven't covered using L'Hospital's)

2. Find the following limits:

(a) $\lim_{x \to 4} \frac{(x+4)\sqrt{x}}{x^2 - 16}$

$= \lim_{x \to 4} \frac{(x+4)\sqrt{x}}{(x+4)(x-4)} = \lim_{x \to 4} \frac{\sqrt{x}}{x-4}$

This limit is undefined

(unless there was a typo in the problem?)

(b) $\lim_{x \to 0} f(x)$, where

$$f(x) = \begin{cases} \sqrt{x} + 2, & x \geq 0 \\ \frac{2 \tan x}{x}, & x < 0 \end{cases}$$

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (\sqrt{x} + 2) = 2$

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{2 \tan x}{x} = 2$

$= \lim_{x \to 0-} 2\sec^2 x$ (by L'Hospital's Rule)

= 2

(Again, let me know if you haven't covered using L'Hospital's)

Therefore $\lim_{x \to 0} f(x) = 2$

3. Use the **definition** of derivative to find the derivative of:

$$f(x)=7x^2-3x-4$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left(7(x+h)^2-3(x+h)-4\right)-\left(7x^2-3x-4\right)}{h}$$

$$=\lim_{h\to 0}\frac{14xh+7h^2-3h}{h}=\lim_{h\to 0}\left(14x+7h-3\right)=14x-3$$

(If you use different notation, e.g. Δx instead of h, let me know)

4. Use the **definition** of derivative to find the derivative of:

$$f(x)=\frac{1-2x}{2x+1}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1-2(x+h)}{2(x+h)+1}-\frac{1-2x}{2x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1-2x-2h}{2x+2h+1}-\frac{1-2x}{2x+1}}{h}=\lim_{h\to 0}\frac{(2x+1)(1-2x-2h)+(2x-1)(2x+2h+1)}{h(2x+2h+1)(2x+1)}$$

$$=\lim_{h\to 0}\frac{\left(-4x^2-2xh+1-2h\right)-\left(2h+1-4x^2-2xh\right)}{h(2x+2h+1)(2x+1)}=\lim_{h\to 0}\frac{-4h}{h(2x+2h+1)(2x+1)}$$

$$=\lim_{h\to 0}\frac{-4}{(2x+2h+1)(2x+1)}=-\frac{4}{(2x+1)^2}$$

5. Differentiate and simplify:

(a) $\sin^2\left(4x^2-5\right)$

$$=2\sin\left(4x^2-5\right)\cos\left(4x^2-5\right)8x=16x\sin\left(4x^2-5\right)\cos\left(4x^2-5\right)$$

$=8x\sin\left(8x^2-10\right)$ (half-angle formula)

(b) $\ln\left(\frac{\sqrt{x^2-1}}{x+1}\right)$

$$=\frac{x+1}{\sqrt{x^2-1}}\,\frac{(x+1)\frac{1}{2}\left(x^2-1\right)^{-1/2}2x-\sqrt{x^2-1}}{(x+1)^2}=\frac{\frac{2x(x+1)}{2\left(x^2-1\right)}-1}{x+1}$$

$$=\frac{2x}{2(x+1)(x-1)}-\frac{1}{x+1}=\frac{2x-2(x-1)}{2(x+1)(x-1)}=\frac{1}{x^2-1}$$

6. Differentiate and simplify:

(a) $\frac{2x}{5-x}$

$$f'(x)=\frac{(5-x)2+2x}{(5-x)^2}=\frac{10}{(5-x)^2}$$

(b) $x^2\sqrt{1-2x^2}$

$$f'(x)=\frac{x^2}{2\sqrt{1-2x^2}}(-4x)+2x\sqrt{1-2x^2}=\frac{-2x^3+2x\left(1-2x^2\right)}{\sqrt{1-2x^2}}=\frac{2x-6x^3}{\sqrt{1-2x^2}}$$

$$=\frac{x^2\sqrt{1-2x^2}+2x-4x^3}{2\left(1-2x^2\right)}$$

7. Find the equation of the tangent line in slope-intercept form of the curve given by:

$\frac{2y}{x}+y^2-5x^2=-2$, passing through $(1,1)$

$$2y+xy^2-5x^3+2x=0$$

$$2dy+2xydy+y^2dx-15x^2dx+2dx=0$$

$$(2+2xy)dy=\left(15x^2-y^2-2\right)dx$$

$$\frac{dy}{dx}=\frac{15x^2-y^2-2}{2+2xy}$$

At $(1,1)$, $\frac{dy}{dx}=3$

$$y=mx+b=3x+b$$

Substituting in $(1,1)$,

$$1=3+b$$

$$b=-2$$

$$y=3x-2$$

8. The height of a ball thrown up from the ground level is given by $h(t) = -5t^2 + 50t$, where h is measured in feet and t is measured in seconds.

(a) How high does the ball go?

$h'(t) = -10t + 50$

$h'(0) = 0$ at $t = 5$ seconds, $h(5) = 125$ feet

125 feet.

(b) How long does it take to return to the ground?

2 x 5 seconds = 10 seconds

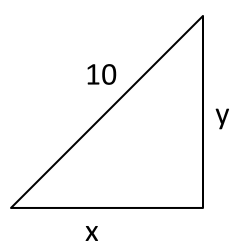
As a double-check, h(10) = 0

(c) What is its velocity just before hitting the ground?

h'(10) = -50

50 feet/second

9. A 10 foot wooden plank leaning against the side of a building is being pulled away so that the base moves away at a rate of 4 ft/sec. How fast is the top of the plank moving down the side of the building when the base of the plank is 6 ft away from the building?



$x^2 + y^2 = 10^2 = 100$

$2xdx + 2ydy = 0$

$\frac{dy}{dx} = -\frac{x}{y}$

$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \left(-\frac{x}{y}\right)(4) = -\frac{4x}{y}$

At $x = 6$, $y = \sqrt{10^2 - 6^2} = 8$, $\frac{dy}{dt} = -\frac{4(6)}{8} = -3$

3 ft/sec

10. A spherical soap bubble is inflated so that its volume is increasing a rate of 2 cubic feet per minute. How fast is the radius of the bubble increasing when the diameter is 1 foot?

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \left(4\pi r^2\right)\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

At r=1/2, $\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi\left(\frac{1}{2}\right)^2}2 = \frac{2}{\pi}$ feet/minute