This assignment covers Block C. It has six questions.

Question 1 – 20 marks

You should be able to answer this question after studying Chapter C1.

This question concerns the function

$$f(x) = x^3 + 3x^2 - 9x + 15.$$

(a) Find the stationary points of this function. [*6*]

(b) (i) Using the strategy to apply the First Derivative Test, classify the left-hand stationary point found in part (a). [*4*]

(ii) Using the Second Derivative Test, classify the right-hand stationary point found in part (a). [*3*]

(c) Find the y-coordinate of each of the stationary points on the graph of the function $f(x)$, and also evaluate $f(0)$. [*3*]

(d) Hence draw a rough sketch of the graph of the function $f(x)$. [*4*]

Question 2 – 15 marks

You should be able to answer this question after studying Chapter C1.

In each of the following parts, you should simplify your answers where it is appropriate to do so.

(a) (i) Write down the derivative of each of the functions

$$f(x) = x^{2/3} \quad (x > 0) \qquad \text{and} \qquad g(x) = \sin(2x).$$ [*2*]

(ii) Hence, by using the Product Rule, differentiate the function

$$k(x) = x^{2/3} \sin(2x) \quad (x > 0).$$ [*2*]

(b) (i) Write down the derivative of each of the functions

$$f(t) = t^4 + 5 \qquad \text{and} \qquad g(t) = \ln(3t) \quad (t > 0).$$ [*2*]

(ii) Hence, by using the Quotient Rule, differentiate the function

$$k(t) = \frac{t^4 + 5}{\ln(3t)} \quad (t > \tfrac{1}{3}).$$ [*3*]

(c) (i) Write down the derivative of the function

$$f(x) = \cos(6x).$$ [*1*]

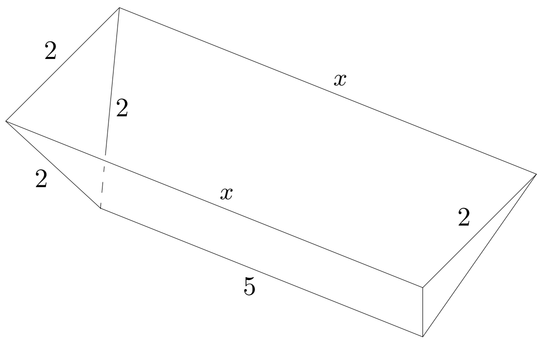
(ii) Hence, by using the Composite Rule, differentiate the function

$$k(x) = e^{\cos(6x)}.$$ [*5*]

Question 3 – 10 marks

You should be able to answer this question after studying Chapter C1.

A container for water has a linear base of length 5 metres and a horizontal rectangular top of length x metres. Each end of the container is an equilateral triangle of side 2 metres. The container, shown in the diagram below, is symmetric about vertical planes through the centre of the rectangular top and parallel to its sides. The value of x is between $5-2\sqrt{3}$ and $5+2\sqrt{3}$. (The container cannot exist for other values of x.)



The total capacity $V(x)\,\mathrm{m}^3$ of the container is given by

$$V(x) = \tfrac{1}{6}(5+2x)\sqrt{12-(x-5)^2} \quad (5-2\sqrt{3} \le x \le 5+2\sqrt{3}).$$

(You are *not* asked to derive this formula.)

For parts (a) and (b) (and for part (c), if you use Mathcad there) you should provide a printout annotated with enough explanation to make it clear what you have done.

NB: If you define x to be a range variable in part (a) and wish to use x in a symbolic calculation in part (b), then you will need to insert the definition $x := x$ between the two parts in your worksheet. (For more details, see the bottom of page 49 in A Guide to Mathcad.*)*

(a) Use Mathcad to obtain the graph of the function $V(x)$. [2]

(b) This part of the question requires the use of Mathcad in each sub-part.

(i) By using the differentiation facility and the symbolic keyword 'simplify', find an expression for the derivative $V'(x)$. [2]

(ii) By either applying a solve block or solving symbolically, find a value of x for which $V'(x) = 0$. [2]

(iii) Verify, by the Second Derivative Test, that this value of x corresponds to a local maximum of $V(x)$. (It should be apparent from the graph obtained in part (a) that this is also an overall maximum within the domain of $V(x)$.) [2]

(c) Using Mathcad, or otherwise, find the maximum possible capacity of the container, according to the model. [2]

Question 4 – 25 marks

You should be able to answer this question after studying Chapter C2.

(a) Find the indefinite integrals of the following functions.

(i) $f(t) = 5\cos(10t) + 12e^{-3t}$ [3]

(ii) $g(x) = \dfrac{5 - 6x^3}{x} \quad (x > 0)$ [4]

(iii) $h(u) = \cos^2(3u)$ [6]

(b) Evaluate $\displaystyle\int_1^3 x(7x^2 + 5)\,dx$. [6]

(c) (i) Write down a definite integral that will give the value of the area under the curve $y = x^2 \cos\left(\frac{1}{5}x\right)$ between $x = \frac{1}{2}\pi$ and $x = \frac{5}{2}\pi$.

(The expression $x^2 \cos\left(\frac{1}{5}x\right)$ takes no negative values for $\frac{1}{2}\pi \leq x \leq \frac{5}{2}\pi$. You are *not* asked to evaluate the integral by hand.) [2]

For part (c)(ii) you should provide a printout of your working.

(ii) Use Mathcad to find the area described in part (c)(i), giving your answer correct to three decimal places. [4]

Question 5 – 10 marks

You should be able to answer this question after studying Chapter C2.

A rocket is modelled by a particle which moves along a vertical line. From launch, the rocket rises until its motor cuts out after 16 seconds. At this time it has reached a height of 560 metres above the launch pad and attained an upward velocity of $90\,\mathrm{m\,s^{-1}}$. From this time on, the rocket has constant upward acceleration $-10\,\mathrm{m\,s^{-2}}$ (due to the effect of gravity alone).

Choose the s-axis (for the position of the particle that represents the rocket) to point upwards, with origin at the launch pad. Take $t = 0$ to be the time when the rocket motor cuts out.

(a) What is the maximum height (above the launch pad) reached by the rocket? [4]

(b) How long (from launch) does the rocket take to reach this maximum height? [2]

(c) After how long (from launch) does the rocket crash onto the launch pad? [4]

Question 6 – 20 marks

You should be able to answer this question after studying Chapter C3.

(a) Solve the initial-value problem

$$\frac{dy}{dx} = \frac{e^{2x}+1}{e^{2x}+2x+2} \quad (x > 0), \qquad y = 3 \text{ when } x = 0.$$

(You may find equation (2.4) in Chapter C2 helpful when integrating.) [6]

(b) (i) Using equation (2.3) in Chapter C2, show that

$$\int \frac{\cos x}{(2+\sin x)^2}\,dx = -\frac{1}{2+\sin x} + c,$$

where c is an arbitrary constant. [2]

(ii) Hence find, in implicit form, the general solution of the differential equation

$$\frac{dy}{dx} = \frac{2y^{3/2}\cos x}{(2+\sin x)^2}.$$ [5]

(iii) Find the corresponding particular solution (in implicit form) that satisfies the initial condition $y = 4$ when $x = 0$. [3]

(iv) Find the explicit form of this particular solution. [2]

(v) What is the value of y given by this particular solution when $x = \frac{1}{3}\pi$? Give your answer to four decimal places. [2]